

3. 线性函数举例.

1. 线性函数: $w = \frac{az+b}{cz+d}$

- ① 保圆性
- ② 保形
- ③ 保交比
- ④ 保对称点

例1. ① $w = i \frac{1-z}{1+z}$ 将单位圆变成上半平面

② $w = \frac{i-z}{i+z}$ 将上半平面变成单位圆

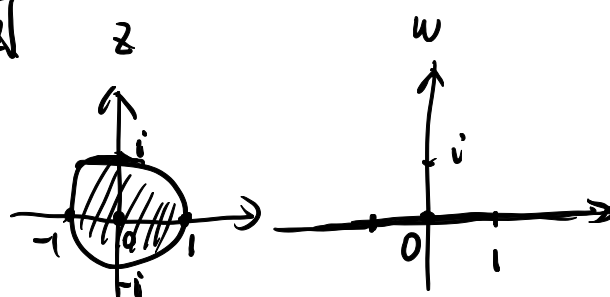
解: ① $w = i \frac{1-z}{1+z}$

$w(1) = 0, w(i) = i \frac{1-i}{1+i}$

$= i \frac{(1-i)^2}{2} = \frac{i(1-i-1)}{2} = 1$

$w(-1) = \infty, w(-i) = i \frac{1+i}{1-i} = -1$

$w(0) = i$



圆内 → 实轴

内部 → 上半平面

⇒ $w = i \frac{1-z}{1+z}$ 将单位圆变成上半平面

② $w = \frac{i-z}{i+z}$ 将上半平面变成单位圆

$f(z) = i \frac{1-z}{1+z}, f^{-1}(z) = ?$

$z = i \frac{1-w}{1+w} \Rightarrow -iz = \frac{1-w}{1+w} \Rightarrow (1+w)(-iz) = 1-w$

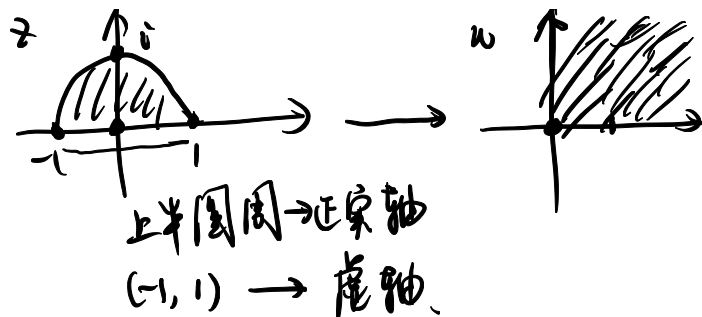
$= -iz - 1 = -w + iwz \Rightarrow w = \frac{-iz-1}{-1+iz} = \frac{i-z}{i+z}$

$f^{-1}(z) = \frac{i-z}{i+z}$ 将上半平面变成单位圆

注: $w = i \frac{1-z}{1+z}$ 也把上半单位圆变成第一象限.

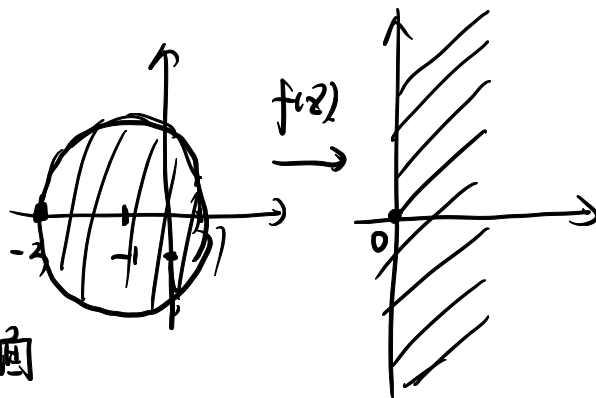
$$w(1)=0, w(i)=1, w(-1)=\infty$$

$$w(0)=i$$



例2. 求分式线性映射数, 将圆心在-1, 半径为2的圆映射成右半平面

解: ① $f(z) = \frac{z+1}{z}$ 圆心-1 $\rightarrow$ 圆心0.
 ② $g(z) = \frac{1-z}{1+z}$ $B(2) \rightarrow B(1)$
 半径为2 $\rightarrow$ 半径为1.



③ $h(z) = i \frac{1-z}{1+z}$ 单位圆 $\rightarrow$ 上半圆面

④ $\varphi(z) = e^{-\frac{\pi i}{2}} z = -iz$ 上半圆面 $\rightarrow$ 右半平面 $e^{-\frac{\pi i}{2}} = -i$

$$\Rightarrow w = \varphi(h(g(f(z)))) = e^{-\frac{\pi i}{2}} \cdot \left(i \frac{1 - \frac{1}{z}(z+1)}{1 + \frac{1}{z}(z+1)} \right) = \frac{2-z-1}{2+z+1} = \frac{1-z}{3+z}$$

$\therefore w = \frac{1-z}{3+z}$ 将圆心在-1, 半径为2的圆变成右半平面.

II. 保交比性. (有顺序的三个点对应, 分式线性映射唯一确定).

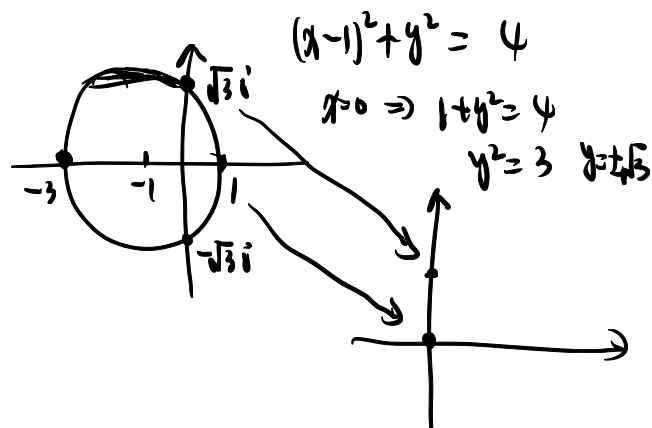
设 $w = \frac{az+b}{cz+d}$

$w(1)=0, w(-3)=\infty, w(\sqrt{3}i)=i$

$$\Rightarrow w(1)=0 = \frac{a+b}{c+d} \Rightarrow a = -b$$

$$w(-3)=\infty = \frac{-3a+b}{-3c+d} \Rightarrow d = 3c$$

$$\Rightarrow w = \frac{az-a}{cz+3c} = \frac{a}{c} \cdot \frac{z-1}{z+3}$$



$$W(\sqrt{3}i) = \frac{a}{c} \frac{\sqrt{3}i - 1}{\sqrt{3}i + 3} = i$$

$$\Rightarrow \frac{a}{c} = i \frac{\sqrt{3}i + 3}{\sqrt{3}i - 1} = i \frac{(\sqrt{3}i + 3)(-1 - \sqrt{3}i)}{(-1 + \sqrt{3}i)(-1 - \sqrt{3}i)} = i \frac{-\sqrt{3}i + 3 - 3 - \sqrt{3}i}{4}$$

$$= i \frac{-4\sqrt{3}i}{4} = \sqrt{3}$$

$$\Rightarrow W = \sqrt{3} \frac{z-1}{z+3}$$

将圆心在-1, 半径为2的圆变成右半平面