调和函数的积分表示、泊松积分公式

1. 泊松 (Poinson) 积分公式：若 $f(z)$ 在 $|z|<R$ 内解析，在 $|z|\le R$ 连续，$f(z)=u(x,y)+iv(x,y)$，$\forall z_0\in|z|<R$

$$\Rightarrow f(z_0)=\frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)}{z-z_0}dz=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})}{Re^{i\theta}-re^{i\varphi}}iRe^{i\theta}d\theta \quad \left(\begin{aligned}z&=Re^{i\theta}\\ z_0&=re^{i\varphi}\end{aligned}\right)$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\theta})\cdot Re^{i\theta}}{Re^{i\theta}-re^{i\varphi}}d\theta \quad ①$$

$\left(z_2 与 z_1 关于 |z-a|=R 对称 \quad z_2-a=\dfrac{R^2}{\overline{z_1-a}}\right)$

设 z_0^* 为 z_0 的对称点，$z^*=\dfrac{R^2}{\bar{z}}=\dfrac{R^2}{re^{i\varphi}}=\dfrac{R^2e^{i\varphi}}{r}$

$$\Rightarrow 0=\frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)}{z-z_0^*}dz$$

（z_0^* 在圆外，$\dfrac{f(z)}{z-z_0^*}$ 解析）。

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})\cdot iRe^{i\theta}}{Re^{i\theta}-\frac{R^2e^{i\varphi}}{r}}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(Re^{i\theta})re^{i\theta}}{re^{i\theta}-Re^{i\varphi}}d\theta \quad ②$$

$$①-② \Rightarrow f(z_0)=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\theta})\left[\frac{Re^{i\theta}}{Re^{i\theta}-re^{i\varphi}}-\frac{re^{i\theta}}{re^{i\theta}-Re^{i\varphi}}\right]d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\theta})\frac{Re^{i\theta}(re^{i\theta}-Re^{i\varphi})-re^{i\theta}(Re^{i\theta}-re^{i\varphi})}{(Re^{i\theta}-re^{i\varphi})(re^{i\theta}-Re^{i\varphi})}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\theta})\frac{Rre^{2i\theta}-R^2e^{i(\theta+\varphi)}-Rre^{2i\theta}+r^2e^{i(\theta+\varphi)}}{Rre^{2i\theta}-r^2e^{i(\varphi+\theta)}-R^2e^{i(\theta+\varphi)}+Rre^{2i\varphi}}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\theta})\frac{(r^2-R^2)e^{i(\theta+\varphi)}}{Rr[e^{2i\theta}+e^{2i\varphi}]-(R^2+r^2)e^{i(\theta+\varphi)}}d\theta$$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(Re^{i\theta})\frac{R^2-r^2}{R^2+r^2-Rr(e^{2i\theta}+e^{2i\varphi})e^{-i(\theta+\varphi)}}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \frac{R^2-r^2}{R^2+r^2-Rr\left[e^{i(\theta-\varphi)}+e^{-i(\theta-\varphi)}\right]}\, d\theta$$

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta}) \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}\, d\theta$$

$$= u(z_0)+iv(z_0)$$

$$\Rightarrow \quad u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(Re^{i\theta}) \frac{R^2-r^2}{R^2+r^2-2Rr\cos(\theta-\varphi)}\, d\theta$$ （泊松积分公式）

2. 定理：任何一个在圆内调和，在闭圆上连续的函数，其在圆内的值都可以用泊松积分公式给出