估计量的评选标准：无偏性、有效性、相合性

1. 无偏性：$E(\hat{\theta})=\theta$

例1. $\bar{x}$ 是总体均值 μ 的无偏估计.

证明：$\bar{x}=\frac{1}{n}\sum\limits_{i=1}^{n}x_i$，$E(\bar{x})=E\left(\frac{1}{n}\sum\limits_{i=1}^{n}x_i\right)=\frac{1}{n}\sum\limits_{i=1}^{n}\underbrace{E(x_i)}_{\mu}=\frac{1}{n}\cdot\sum\limits_{i=1}^{n}\mu=\mu$

例2. 样本方差 $S^2=\frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i-\bar{x})^2$ 是总体方差 σ^2 的无偏估计

证明：$S^2=\frac{1}{n-1}\sum\limits_{i=1}^{n}(x_i-\bar{x})=\frac{1}{n-1}\left(\sum x_i^2-n\bar{x}^2\right)$

$$\Rightarrow E(S^2)=\frac{1}{n-1}\left(\sum_{i=1}^{n}E(x_i^2)-nE(\bar{x}^2)\right)$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}(\sigma^2+\mu^2)-n\cdot\left(D(\bar{x})+(E(\bar{x}))^2\right)\right)$$

$$=\frac{1}{n-1}\left(n\sigma^2+n\mu^2-n\cdot\left(\frac{\sigma^2}{n}+\mu^2\right)\right)$$

$$=\frac{1}{n-1}(n-1)\sigma^2=\sigma^2$$

$$\sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{i=1}^{n}(x_i^2-2x_i\bar{x}+\bar{x}^2)$$

$$=\sum_{i=1}^{n}x_i^2-2\bar{x}\cdot\sum x_i+n\bar{x}^2$$

$$=\sum_{i=1}^{n}x_i^2-2\cdot\frac{1}{n}\sum x_i\cdot\sum x_i+n\bar{x}^2$$

$$=\sum_{i=1}^{n}x_i^2-2n\cdot\left(\frac{1}{n}\sum x_i\right)\left(\frac{1}{n}\sum x_i\right)+n\bar{x}^2$$

$$=\sum_{i=1}^{n}x_i^2-n\bar{x}^2$$

$$\boxed{\begin{array}{l}D(X)=E(X^2)-(E(X))^2\\ \Rightarrow E(X^2)=DX+(E(X))^2\end{array}}$$

2. 有效性：$D(\hat{\theta}_1)\le D(\hat{\theta}_2)$ 称 $\hat{\theta}_1$ 比 $\hat{\theta}_2$ 更有效.

例3. 设 x_1,x_2,x_3 为来自于均值为 μ，方差为 σ^2 的总体的一组样本. $\hat{\theta}_1=\bar{x}=\frac{1}{3}(x_1+x_2+x_3)$，$\hat{\theta}_2=\frac{1}{2}x_1+\frac{1}{3}x_2+\frac{1}{6}x_3$. 试证明 $\hat{\theta}_1$ 和 $\hat{\theta}_2$ 都是 μ 的无偏估计并确定哪一个估计量更有效

证明：$E(\hat{\theta}_1)=E\left(\frac{1}{3}(x_1+x_2+x_3)\right)=\frac{1}{3}\left(E(x_1)+E(x_2)+E(x_3)\right)=\frac{1}{3}\cdot3\mu=\mu$

$E(\hat{\theta}_2)=E\left(\frac{1}{2}x_1+\frac{1}{3}x_2+\frac{1}{6}x_3\right)=\frac{1}{2}E(x_1)+\frac{1}{3}E(x_2)+\frac{1}{6}E(x_3)=\frac{1}{2}\mu+\frac{1}{3}\mu+\frac{1}{6}\mu=\mu$

$\Rightarrow$ $\hat{\theta}_1$ 和 $\hat{\theta}_2$ 都是 μ 的无偏估计.

$$D(\hat{\theta}_1)=D(\bar{x})=D\left(\frac{1}{3}(x_1+x_2+x_3)\right)=\frac{1}{9}(Dx_1+Dx_2+Dx_3)=\frac{1}{9}\cdot(\sigma^2+\sigma^2+\sigma^2)$$

$$=\frac{1}{3}\sigma^2$$

$$D(\hat{\theta}_2) = D\left(\frac{1}{2}x_1+\frac{1}{3}x_2+\frac{1}{6}x_3\right) = \frac{1}{4}\sigma^2+\frac{1}{9}\sigma^2+\frac{1}{36}\sigma^2 = \frac{14}{36}\sigma^2$$

$\because D(\hat{\theta}_1) < D(\hat{\theta}_2) \Rightarrow \hat{\theta}_1$ 比 $\hat{\theta}_2$ 更有效.

3. 相合性（一致性）: $\lim\limits_{n\to\infty} P(|\hat{\theta}-\theta| < \varepsilon) = \underline{\underline{1}}$ （$\hat{\theta}$ 依概率收敛于 θ）

例4. $\bar{x}$ 是总体均值 μ 的相合估计.

证明:（切比雪夫大数定理: $P\{|\bar{X}-\underline{\underline{\mu}}| < \varepsilon\} \geq 1-\frac{D\bar{X}}{\varepsilon^2}$ $\quad P\{|\bar{X}-E(\bar{X})| < \varepsilon\} \geq 1-\frac{D\bar{X}}{\varepsilon^2}$）

$P\{|\bar{X}-\mu| > \varepsilon\} \leq \frac{D\bar{X}}{\varepsilon^2}$,

$$P\{|\bar{x}-E(\bar{x})| < \varepsilon\} \geq 1-\frac{D\bar{x}}{\varepsilon^2} \qquad \because E(\bar{x}) = \mu$$

$$\Rightarrow P\{|\bar{x}-\mu| < \varepsilon\} \geq 1-\frac{D\bar{x}}{\varepsilon^2} = 1-\frac{1}{\varepsilon^2}\cdot\frac{\sigma^2}{\underline{\underline{n}}}$$

$$\Rightarrow \lim_{n\to\infty} P\{|\bar{x}-\mu| < \varepsilon\} \geq 1 \Rightarrow \lim_{n\to\infty} P\{|\bar{x}-\mu| < \varepsilon\} = 1$$

定理: 若 $\lim\limits_{n\to\infty} E(\hat{\theta}) = \theta$, $\underline{\lim\limits_{n\to\infty} D(\hat{\theta}) = 0} \Rightarrow \hat{\theta}$ 是 θ 的相合性估计.

$$\begin{cases} D(\bar{x}) = \frac{\sigma^2}{n} \\ E(\bar{x}) = \mu \end{cases} \qquad \lim_{n\to\infty} D(\bar{x}) = 0 \qquad \therefore \bar{x} \text{ 是 } \mu \text{ 的相合性估计.}$$