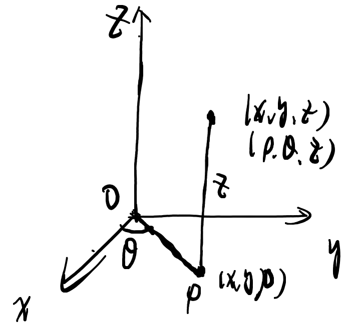
柱坐标下计算三重积分

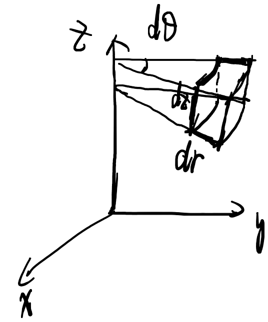
1、柱坐标



$$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases} \qquad \begin{cases} \rho=\sqrt{x^2+y^2} \\ \theta=\begin{cases}\arctan\frac{y}{x} & \text{I、II} \\ \arctan\frac{y}{x}+\pi & \text{III、IV}\end{cases} \\ z=z \end{cases}$$

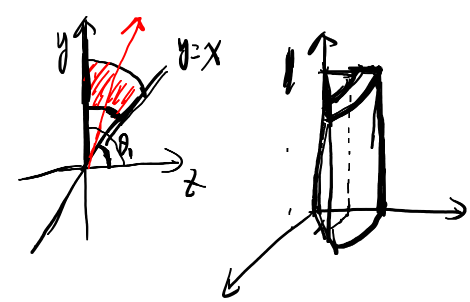
2. $$\iiint_V f(x,y,z)\,dV = \iiint_V f(\rho\cos\theta,\rho\sin\theta,z)\cdot\rho\,d\rho\,d\theta\,dz$$

$$dV=dA\cdot dz=\rho\,d\rho\,d\theta\,dz$$



3、适用范围：区域在xoy平面上投影为圆

例1. 计算 $\iiint_V (x^2+y^2)dV$，其中V是由 $x^2+y^2=1$，$x^2+y^2=4$，$z=0$，$z=1$，$x=0$，$x=y$ 所围成在第一卦限部分。



解：$V=\{(\rho,\theta,z)\mid 1\le\rho\le 2,\ \frac{\pi}{4}\le\theta\le\frac{\pi}{2},\ 0\le z\le 1\}$

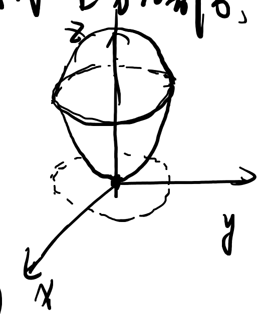
$$\iiint_V (x^2+y^2)dV=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^2\int_0^1 \rho^2\rho\,dz\,d\rho\,d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^2 \rho^3\cdot z\Big|_0^1 d\rho\,d\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_1^2\rho^3 d\rho\,d\theta$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{4}\rho^4\Big|_1^2 d\theta=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(4-\frac{1}{4}\right)d\theta=\frac{15}{4}\cdot\theta\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=\frac{15}{16}\pi$$

例2. 计算立体的体积，其中立体是位于 $x^2+y^2+z^2=6$ 的下方而在 $z=x^2+y^2$ 上方的部分。



解：$V=\iiint_V dV$　　交线：$\begin{cases}x^2+y^2=z\\x^2+y^2+z^2=6\end{cases}$ $\Rightarrow z^2=6-z$ $\Rightarrow z^2+z-6=0$

$(z+3)(z-2)=0$ $\Rightarrow z=2$（$z=-3$舍去）

$V=\{(\rho,\theta,z)\mid 0\le\theta\le 2\pi,\ 0\le\rho\le\sqrt{2},\ \rho^2\le z\le\sqrt{6-\rho^2}\}$

$\Rightarrow \begin{cases}x^2+y^2=2\\z=2\end{cases}$

$\because$ $x^2+y^2=\rho^2$, $x^2+y^2=z \Rightarrow z=\rho^2$

$x^2+y^2+z^2=6 \Rightarrow \rho^2+z^2=6 \Rightarrow z=\pm\sqrt{6-\rho^2}$ $(z>0)$

$\Rightarrow z=\sqrt{6-\rho^2}$

$$\Rightarrow V=\iiint_V dv=\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{\rho^2}^{\sqrt{6-\rho^2}}\rho\, dz\, d\rho\, d\theta$$

$$=\int_0^{2\pi}\int_0^{\sqrt{2}}\rho\cdot z\Big|_{\rho^2}^{\sqrt{6-\rho^2}} d\rho\, d\theta=\int_0^{2\pi}\int_0^{\sqrt{2}}\left(\rho\sqrt{6-\rho^2}-\rho^3\right)d\rho\, d\theta$$

$u=6-\rho^2$

$du=-2\rho d\rho$

$$=\int_0^{2\pi}\left[(6-\rho^2)^{\frac{3}{2}}\cdot\frac{2}{3}\left(-\frac{1}{2}\right)-\frac{1}{4}\rho^4\right]_0^{\sqrt{2}}d\theta$$

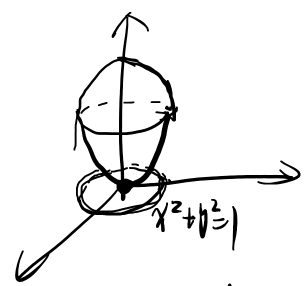
$$=\int_0^{2\pi}\left[8\cdot\left(-\frac{1}{3}\right)-1\right]-\left(-\frac{1}{3}6^{\frac{3}{2}}-0\right)d\theta$$

$$=\int_0^{2\pi}\left[\frac{6^{\frac{3}{2}}}{3}-\frac{11}{3}\right]d\theta=\left(\frac{6\sqrt{6}}{3}-\frac{11}{3}\right)\theta\Big|_0^{2\pi}$$

$$=2\pi\left(2\sqrt{6}-\frac{11}{3}\right)$$

例3. $\iiint_V z\, dv$，其中 V 是由曲面 $z=\sqrt{2-x^2-y^2}$ 及 $z=x^2+y^2$ 所围成

$V=\{(\rho,\theta,z) \mid 0\le\theta\le 2\pi, 0\le\rho\le 1, \rho^2\le z\le\sqrt{2-\rho^2}\}$



$$\begin{cases} z=\sqrt{2-x^2-y^2} \\ z=x^2+y^2 \end{cases} \Rightarrow \begin{cases} x^2+y^2+z^2=2 \\ x^2+y^2=z \end{cases} \Rightarrow z^2=2-z$$

$z^2+z-2=0 \Rightarrow \begin{cases} z=1 & (z=-2 \text{ 舍}) \\ z=x^2+y^2 \end{cases}$

$\Rightarrow x^2+y^2=1$

$$\Rightarrow \iiint_V z\, dv=\int_0^{2\pi}\int_0^1\int_{\rho^2}^{\sqrt{2-\rho^2}} z\cdot\rho\, dz\, d\rho\, d\theta$$

$$=\int_0^{2\pi}\int_0^1\frac{1}{2}\rho z^2\Big|_{\rho^2}^{\sqrt{2-\rho^2}}d\rho\, d\theta$$

$$=\int_0^{2\pi}\int_0^1\frac{1}{2}\rho(2-\rho^2-\rho^4)\, d\rho\, d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left(\rho^2-\frac{1}{4}\rho^4-\frac{1}{6}\rho^6\right)\Big|_0^1=\frac{1}{2}\cdot 2\pi\cdot\left(1-\frac{1}{4}-\frac{1}{6}\right)$$

$$=\pi\left(\frac{7}{12}\right)=\frac{7}{12}\pi$$