实对称矩阵的对角化.

1. 定理：A是实对称矩阵，则存在正交矩阵 Q，使得 $Q^{-1}AQ=D$

2. 实对称矩阵对角化步骤

① 特征值

② 特征向量

* ③ 特征向量单位正交化 $\Rightarrow \vec{e_i}$（同一个特征值的特征向量正交化，再单位化）

④ $Q=(\vec{e_1}\ \vec{e_2}\ \cdots\ \vec{e_n})$，$D=\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}$

例1. $A=\begin{pmatrix}4&2&2\\2&4&2\\2&2&4\end{pmatrix}$，求正交矩阵Q，使得 $Q^{-1}AQ$ $(Q^{-1}=Q^T)$

解：① $|A-\lambda I|=\begin{vmatrix}4-\lambda&2&2\\2&4-\lambda&2\\2&2&4-\lambda\end{vmatrix}=\begin{vmatrix}8-\lambda&8-\lambda&8-\lambda\\2&4-\lambda&2\\2&2&4-\lambda\end{vmatrix}=(8-\lambda)\begin{vmatrix}1&1&1\\2&4-\lambda&2\\2&2&(4-\lambda)\end{vmatrix}$

$=(8-\lambda)\begin{vmatrix}1&1&1\\0&2-\lambda&0\\0&0&2-\lambda\end{vmatrix}=(8-\lambda)(2-\lambda)^2 \Rightarrow \lambda_1=8,\ \lambda_{2,3}=2$

② $\lambda_1=8$ 时，$(A-\lambda I)=\begin{pmatrix}-4&2&2\\2&-4&2\\2&2&-4\end{pmatrix}\sim\begin{pmatrix}-4&2&2\\2&-4&2\\0&0&0\end{pmatrix}$

$\sim\begin{pmatrix}2&-4&2\\-4&2&2\\0&0&0\end{pmatrix}\sim\begin{pmatrix}2&-4&2\\0&-6&6\\ \end{pmatrix}\sim\begin{pmatrix}1&-2&1\\0&1&-1\end{pmatrix}$

$\sim\begin{pmatrix}1&0&-1\\0&1&-1\end{pmatrix}$ $\quad \vec{x_1}=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $\quad \vec{e_1}=\begin{pmatrix}\frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\\ \frac{1}{\sqrt3}\end{pmatrix}$

$\lambda_{2,3}=2$ 时 $A-\lambda I=\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}\sim\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$

$\vec{x_2}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$，$\vec{x_3}=\begin{pmatrix}0\\-1\\1\end{pmatrix}$

$$\vec{b}_2 = \vec{x}_2 , \qquad \vec{b}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{b}_2}{\|\vec{b}_2\|^2}\vec{b}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix} \qquad \|\vec{b}_3\| = \sqrt{\frac{3}{2}}$$

$$\vec{e}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} , \qquad \vec{e}_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

(3) $$Q = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix} , \qquad D = \begin{pmatrix} 8 & & \\ & 2 & \\ & & 2 \end{pmatrix}$$

$$Q^{-1} = Q^{T} = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$